

# Singular behaviour on folding path characterised by rigid foldability analysis

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## Abstract

A rigid origami model is useful in studying the kinematic behaviour of deployable structures, which has posed interesting problems from a mathematical standpoint. Previous searches have shown that the mode of rigid folding angle motion can be extracted by considering the time derivative of constraint conditions around a single vertex for a flat disk surface, the sum of whose plane angle is equal to  $2\pi$ . The procedure is described in the following.

Around a single vertex in which  $n$ -fold lines are concentrated, formed by plane angles  $\theta_1, \theta_2, \dots, \theta_n$ , and folding angles  $\rho_1, \rho_2, \dots, \rho_n$ , the constraint condition can be expressed as reported in eq. (1) by using rotational matrices (fig. 1).

$$\mathbf{R}(\boldsymbol{\rho}) = \boldsymbol{\chi}_{0,1} \boldsymbol{\chi}_{1,2} \cdots \boldsymbol{\chi}_{n-1,n} = \mathbf{I} \quad (1)$$

where

$$\boldsymbol{\chi}_{i-1,i} = \mathbf{Y}\mathbf{P} = \begin{bmatrix} \cos \theta_{i-1,i} & -\sin \theta_{i-1,i} & 0 \\ \sin \theta_{i-1,i} & \cos \theta_{i-1,i} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \rho_i & -\sin \rho_i \\ 0 & \sin \rho_i & \cos \rho_i \end{bmatrix}. \quad (2)$$

Given that this condition (1) is also satisfied after a time period of  $\Delta t$ , eq. (3) can follow.

$$\begin{aligned} \mathbf{R}(\rho_1, \rho_2, \dots)|_{t=0} &+ \left[ \sum_{i=1}^n \frac{\partial \mathbf{R}}{\partial \rho_i} \dot{\rho}_i \right]_{t=0} \Delta t \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \frac{\partial \mathbf{R}}{\partial \rho_i} \ddot{\rho}_i + \sum_i \sum_j \frac{\partial^2 \mathbf{R}}{\partial \rho_i \partial \rho_j} \dot{\rho}_i \dot{\rho}_j \right]_{t=0} \Delta t^2 + o(\Delta t^3) = \mathbf{I} \end{aligned} \quad (3)$$

Considering that the first derivative term is equal to zero, eq. (4) can be obtained.

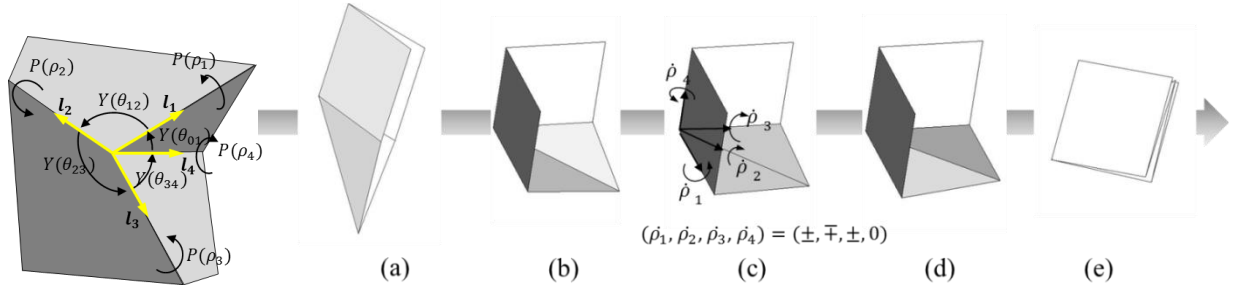
$$\mathbf{C}\dot{\boldsymbol{\rho}} = [\mathbf{l}_1 \mathbf{l}_2 \cdots \mathbf{l}_n] \begin{bmatrix} \dot{\rho}_1 \\ \dot{\rho}_2 \\ \vdots \\ \dot{\rho}_n \end{bmatrix} = \sum_{i=1}^n \dot{\rho}_i \mathbf{l}_i = \mathbf{0} \quad (4)$$

Here, the vector  $\mathbf{l}_i$  represents each of the direction cosines of corresponding fold lines. Moreover, eq. (4) represents the following geometric condition (1<sup>st</sup> condition): "the sum of fold-line unit vectors  $\mathbf{l}_i$ , weighted by  $\dot{\rho}_i$ , should be equal to zero". If all folding angles  $\rho_i$  are equal to zero, the 1<sup>st</sup> condition alone is not sufficient to completely describe the system. In this case, the second derivative term in eq. (3) is required. Such term corresponds to the geometric con-

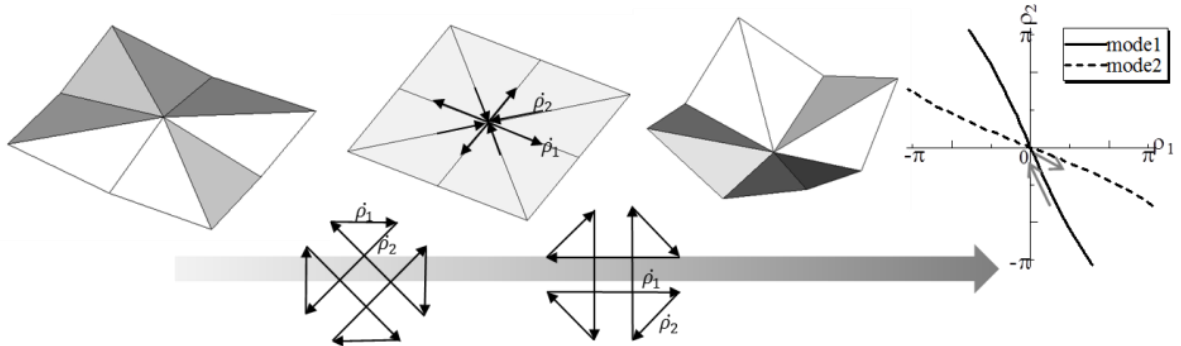
dition (2<sup>nd</sup> condition): "while proceeding in anticlockwise order around the vertex, the operation of connecting each vector  $\dot{\rho}_i \mathbf{l}_i$  to the next one, head-to-tail, allows the realization of a closed loop". In this study, we discuss the relationship between the two presented conditions concerning the rigid-foldable mode of rotation around the hinge-line and the characteristics of folding motion paths.

First, we apply the 1<sup>st</sup> and the 2<sup>nd</sup> conditions to a non-flat disk surface, and compare this case with the one in which a flat disk surface is used. In case of the process of deploying, it is sufficient to apply the 1<sup>st</sup> condition only, whereas if the non-flat disk surface is folded up flatly, both conditions need to be applied, as in the case of a flat disk. However, there also exists the case in which the rotational motion around a particular fold-line cannot be determined by applying the 1<sup>st</sup> condition to the deformation path. As shown in fig. 2, when the 1<sup>st</sup> condition is applied (state (c)), the parameter  $\dot{\rho}_4$  is required to be zero. This is motivated by the fact that  $\rho_4$  is at its minimum at this point of the deformation path. Moreover, this situation is a feature of the non-flat disk surface, which has fold-lines on which the range of the dihedral angle is restricted.

Next, we characterise the non-smooth deformation path for flat disk surfaces by considering the 1<sup>st</sup> and the 2<sup>nd</sup> conditions. As an example, we can observe the non-smooth deformation path from bottom to top, even with the same mountain–valley (M–V) assignment, with respect to a water-bomb crease pattern (fig. 3). In the vicinity of the flat state, it is possible to draw two different types of diagrams by applying the 2<sup>nd</sup> condition. This characteristic about the mode of fold-line rotation corresponds to the singular behaviour around the flat state. Finally, we discuss a method for designing crease patterns by considering the folding motion.



**Figure 1:** Constraint conditions **Figure 2:** Application of conditions to non-flat disk surfaces



**Figure 3:** Example of a non-smooth deformation path